FLEXIBLE CASE ANALYSIS

FOR COMPRESSIBLE

SOLID PROPELLANT GRAIN MOTORS

to

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

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1. INTRODUCTION

Since the propellant grain must support itself, it is a structural member. Moreover, if the grain cracks before or during firing, then there results an unknown thrust-time profile and perhaps a catastropic failure. Part of the design process must therefore be an investigation of the structural integrity of the grain. A fundamental part of this investigation is a stress-strain analysis of the propellant grain. On the basis of the thermal, pressure, and acceleration loads and the support conditions, the stress and strain distribution in the grain is to be determined.

Due to the complex grain geometries and complex loads and supports, a numerical solution must be sought. A numerical method of sufficient versatility to handle these complexities is the finite element method. In particular for continuum problems, the stiffness version of the finite element method has proven to be the most successful.

2. STIFFNESS METHOD OUTLINE

The essential feature of the stiffness method is the ability to analyze small elementary regions of the propellant grain independent of the total configuration and then combine these regions together to find the response of the total configuration.

The stiffness method has similarities to the Ritz method. The equilibrium equations and stress boundary conditions are replaced by an equivalent variational principle, $\delta \pi(\underset{\sim}{u}) = 0$. The grain is divided into elements within which the displacement has approximately a specified form. For a plane strain problem and an axisymmetric elasticity problem, triangular rings are used; in the axially symmetric shell problems, line elements are used. The unknowns are the displacements in the elements. They are expressed in terms of the node displacements which are then determined so that the functional $\pi(\underset{\sim}{u})$ is an extremum.

While the theory underlying the stiffness method is sufficiently general to include three-dimensional problems, the programing and computer capacity needed to carry out practical calculations is not generally available. The current practice is to utilize two-dimensional programs. The plane strain program and the axisymmetric program together provide information about the state of stress which may be used for design. The programs of the Mathematical Sciences Corporation have the ability to include the effect of a flexible case bounding the grain.

3. PLANE STRAIN PROBLEM BOUNDED BY A FLEXIBLE CASE

The plane strain program provides a stiffness method solution to the following problem. In Figure 1 is a finite two-dimensional region bounded by a circular shell.
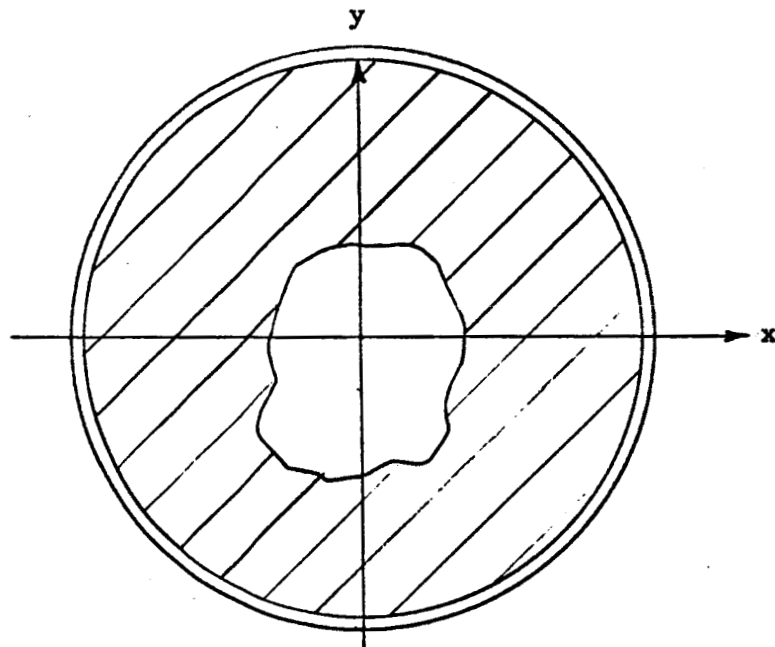

Figure 1

This is the typical situation encountered in a solid propellant grain analysis.

The solid region is governed by the following system of equations, referred to rectangular Cartesian coordinates.

Equilibrium: $\qquad \sigma^{\alpha\beta}{}_{,\beta} + f^{\alpha} = 0$, $\qquad\qquad$ (1)

Strain-Displacement: $\epsilon_{\alpha\beta} = \frac{1}{2}(u_{\alpha,\beta} + u_{\beta,\alpha})$, $\qquad$ (2)

Stress-Strain: $\quad \sigma^{\alpha\beta} = c^{\alpha\beta\gamma\delta}\epsilon_{\gamma\delta} - a^{\alpha\beta}T$, $\quad$ (3)

where

$\sigma^{\alpha\beta}$ $\qquad$ is the stress tensor,

$\epsilon_{\alpha\beta}$ $\qquad$ is the strain tensor,

u_α $\qquad$ is the displacement vector,

$c^{\alpha\beta\gamma\delta}$ $\qquad$ are the coefficients of elasticity,

$a^{\alpha\beta}$ $\qquad$ are the coefficients of thermal stress,

T $\qquad$ is the temperature change,

f^α $\qquad$ is the body force vector.

The comma denotes partial differentiation, and the repeated indices are summed over the range 1, 2. The index 1 is associated with the x direction; the index 2 with the y direction.

At each point of the boundary of the solid with unit normal n_α, either the displacement vector u_α or the surface force

$$F^\alpha = \sigma^{\alpha\beta}n_\beta \qquad (4)$$

must be given. The temperature and body forces must be given throughout the body. The coefficients of elasticity have the following symmetry:

$$c^{\alpha\beta\gamma\delta} = c^{\alpha\beta\delta\gamma} = c^{\beta\alpha\delta\gamma} = c^{\delta\gamma\beta\alpha} .$$

The coefficients $a^{\alpha\beta}$ are symmetric.

The shell is governed by the following equations.

$$\text{Equilibrium:} \quad \left\{ \begin{array}{l} \dfrac{dN}{dx} = 0 \quad , \\[2em] \dfrac{N}{a} + \dfrac{dQ}{dx} + p = 0 \quad , \\[2em] q = \dfrac{dM}{dx} \quad , \end{array} \right. \tag{5}$$

$$\text{Strain-Displacement:} \quad \in \; = \dfrac{dv}{dx} + \dfrac{w}{a} \quad , \tag{6}$$

$$\text{Stress-Strain:} \quad \left\{ \begin{array}{l} N = D \in \; + N_o \quad , \\[2em] M = K \dfrac{d^2 w}{dx^2} \quad , \end{array} \right. \tag{7}$$

where

N is the in-plane stress resultant,

Q is the transverse shear,

M is the moment resultant,

p is the shell pressure distribution,

D is the shell membrane stiffness,

K is the shell bending stiffness,

N_o is the thermal stress resultant,

w is the normal displacement,

v is the tangential displacement,

a is the radius of the shell,

x is a coordinate tangent to the shell.

The stiffness method is applied to the solid using triangular elements and linear variation of displacement in each element. In Figure 2 is a typical element.
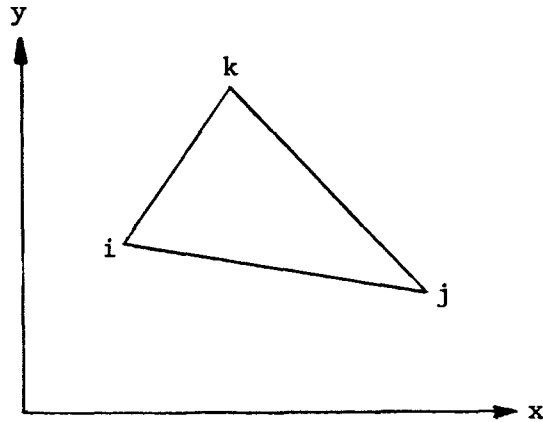

Figure 2

Within the element the displacements u and v are assumed to be of the form

$$u = ax + by + c \quad,$$
$$v = dx + ey + f \quad.$$

(8)

The six constants may be expressed in terms of the displacement at the three nodes i, j, k ;

$$u = f_1(x,y)u_i + f_2(x,y)u_j + f_3(x,y)u_k \quad,$$
$$v = f_4(x,y)v_i + f_5(x,y)v_j + f_6(x,y)v_k \quad.$$

(9)

Note that the indices i, j, k are not free indices here. They denote definite node numbers. This process is repeated for all the elements.

The result is a polyhedral displacement field defined by the values u_i, v_i, . . . at nodes. Putting this expression in the potential energy expression and minimizing results in 2n linear equations in the 2n unknown displacements at the n nodes; the coefficients are the stiffness matrix for the solid.

The stiffness matrix is applied to the shell using line elements and the exact solution to the equations (5) through (7) for concentrated loads at the ends. In Figure 3 is a typical shell element.
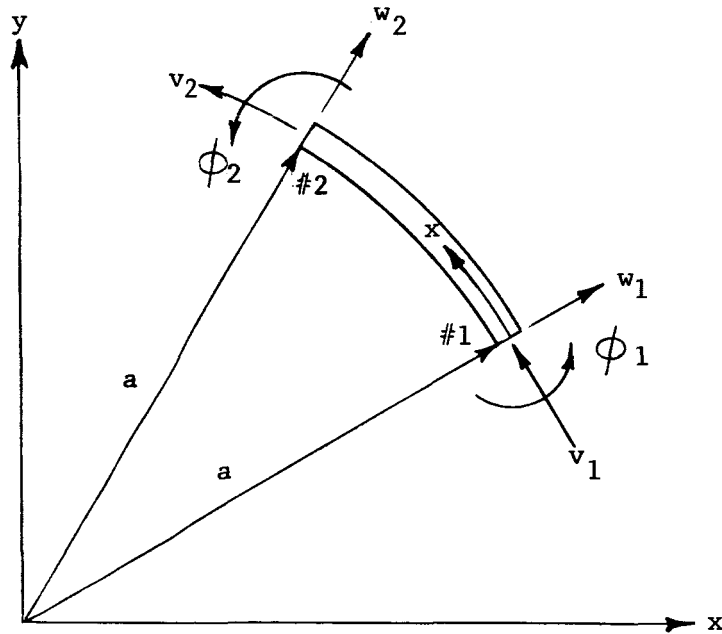

Figure 3

The result is an element stiffness expression,

$$
\begin{bmatrix} k \end{bmatrix} \begin{bmatrix} w_1 \\ v_1 \\ \phi_1 \\ w_2 \\ v_2 \\ \phi_2 \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \\ F_6 \end{bmatrix} \quad , \tag{10}
$$

where the loads F_1 to F_6 are associated with the six displacements; $\begin{bmatrix} k \end{bmatrix}$ is a 6 x 6 element stiffness matrix. The process is repeated for the whole shell. The element stiffnesses are summed together to obtain a stiffness matrix $\begin{bmatrix} k \end{bmatrix}$ for the complete shell.

Theoretically the stiffness matrix for the shell may be combined with that of the solid and inverted to obtain the displacement solution. However, in practice the result is a very poorly conditioned matrix that is difficult to invert. Instead the solution is obtained by an iterative scheme.

The solution is obtained for the solid using zero values for shell displacements, and the reactions of the solid on the shell are calculated. The deflections of the shell are then determined using these reactions as loads. The solution is next obtained for the solid using the corrected values of the shell displacement to obtain new reactions. The procedure is then repeated until the error is as small as desired.

The computer program used to carry out the numerical computations can handle up to 280 nodal points in a triangular breakdown of the solid and 21 nodal points for the shell. The program accounts for body force, temperature, and surface traction loads for the solid and temperature and surface traction loads for the shell. The material properties may be orthotropic and nonhomogeneous for both the shell and the solid.

Using the given loads, nodal point coordinates, and the node groups bounding the elements, the program generates the stiffness matrix for the solid and the shell, solves the system of linear equations to find the displacement solution, and calculates the stresses.

4. AXISYMMETRIC PROBLEM BOUNDED BY A FLEXIBLE CASE

The axisymmetric program provides a stiffness method solution to the following problem. In Figure 4 is a cross section of an axisymmetric body bounded by a flexible case.
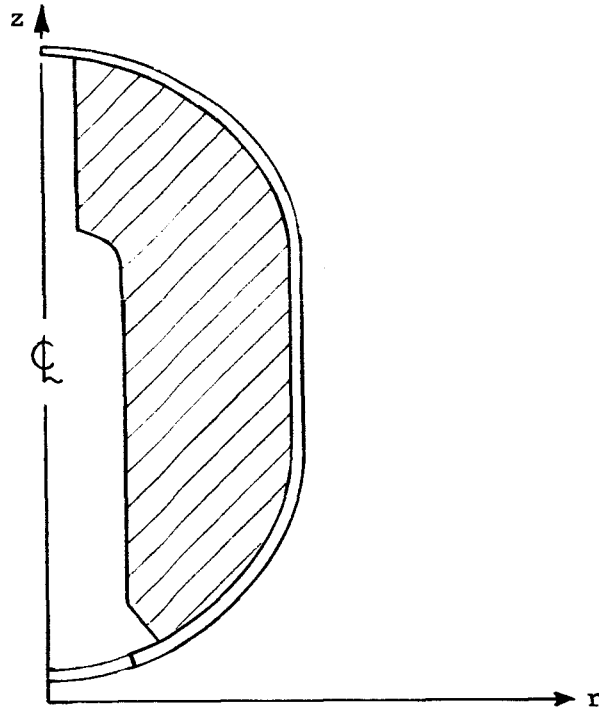

Figure 4

This represents a problem often encountered in solid propellant grain analysis.

The solid region is governed by the following system of equations.

Equilibrium: $\quad \sigma^{ij}{}_{,j} + f^i = 0$, $\qquad (11)$

Strain-Displacement: $\quad \epsilon_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i})$, $\qquad (12)$

Stress-Strain: $\qquad \sigma^{ij} = C^{ijrs} \epsilon_{rs} - a^{rs} T \quad , \qquad$ (13)

where the definition of the terms is the same as before. The comma denotes covariant differentiation in the cylindrical coordinate system r, θ, z ; the indices range from 1 to 3. In this axisymmetric problem, there is no θ dependence in any of the variables, and only the displacements u and w in the r and z directions respectively are nonzero. Circumferential or hoop strains may exist:

$$\epsilon_{22} = \epsilon_{\theta\theta} = \frac{u}{r}$$ (14)

is in general nonzero.

At each point of the boundary (of the solid with unit normal n_i) either the displacement vector u_i or the surface force

$$F^i = \sigma^{ij} n_j$$ (15)

must be specified. Both $u_2 = v$ and $F^2 = F^{\theta}$ must be zero for this axisymmetric problem. The temperature and body forces must be given throughout the body. The coefficients of elasticity have the same symmetry as before. The axisymmetric shell is first approximated by conical elements. In Figure 5 is a two-element approximation.
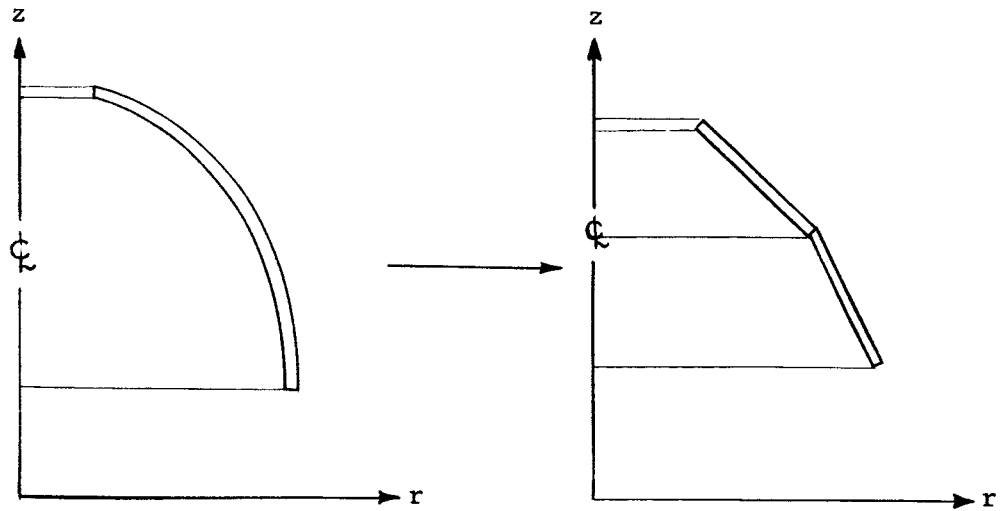
Figure 5

The conical frustrums are governed by the equations obtained from the variational principle

$$\delta \pi = 0 \quad ,$$

where

$$\pi(u,w) = \int_0^\ell \tfrac{1}{2}(C^{\alpha\beta}\epsilon_\alpha \epsilon_\beta + D^{\alpha\beta}\chi_\alpha \chi_\beta)r\,ds$$

$$- \int_0^\ell (T \beta^\alpha \epsilon_\alpha + p^\alpha u_\alpha)r\,ds \quad ; \tag{16}$$

α and β range over 1 and 2, and repeated indices are summed.

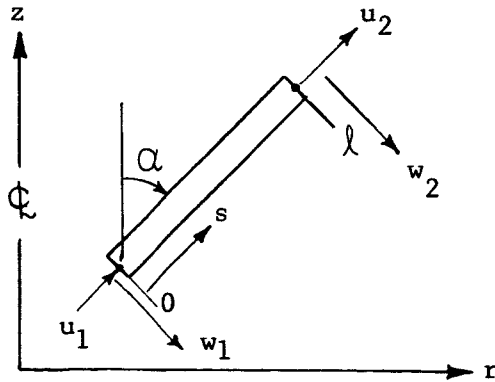Figure 6

Using the notation of Figure 6, the strains ϵ_α and χ_α are given by

$$\epsilon_1 = \frac{du}{ds} \quad , \qquad \epsilon_2 = \frac{u \sin \alpha}{r} + \frac{w \cos \alpha}{r} \quad , \qquad (17)$$

$$\chi_1 = \frac{d^2 w}{ds^2} \quad , \qquad \chi_2 = \frac{\sin \alpha}{r} \frac{dw}{ds} \quad . \qquad (18)$$

The other quantities are defined as follows:

$c^{\alpha\beta}$ = the membrane coefficients of elasticity,

$D^{\alpha\beta}$ = the bending coefficients of elasticity,

$T\beta^\alpha$ = the thermal stresses, and

p^α = the surface tractions.

For the stiffness method, displacements of the form

$$u = (1 - \frac{s}{\ell})u_1 + \frac{s}{\ell} u_2 \qquad (19)$$

and

$$w = \left[1 - 3\left(\frac{s}{\ell}\right)^2 + 2\left(\frac{s}{\ell}\right)^3 \right] w_1 + \left[3\left(\frac{s}{\ell}\right)^2 - 2\left(\frac{s}{\ell}\right)^3 \right] w_2$$

$$+ \left[\frac{s}{\ell} - 2\left(\frac{s}{\ell}\right)^2 + \left(\frac{s}{\ell}\right)^3 \right] \ell w_1{}' + \left[\left(\frac{s}{\ell}\right)^3 - \left(\frac{s}{\ell}\right)^2 \right] \ell w_2{}' \qquad (20)$$

are assumed for each element. From this and equation (16) the element stiffness matrices may be obtained.

As in the plane strain program the shell and solid stiffness matrices are not combined. The solution is obtained by the iteration technique previously described.

The computer program used to carry out the numerical computations can handle up to 280 nodal points in a triangular breakdown of the solid and 100 nodal points for the shell. The program accounts for body forces, temperature, and surface traction loads for the solid and temperature and surface traction loads for the shell. The material properties may be orthotropic and nonhomogeneous for both the shell and the solid.

Using the given loads, nodal point coordinates, and the node groups bounding the elements, the program generates the stiffness matrix for the solid and the shell, solves the system of linear equations to find the displacement solution, and calculates the stresses.

5. REVIEW OF CURRENT STATUS

The programs as they now stand are satisfactory for the majority of problems encountered. However, they cannot handle materials that are incompressible or nearly incompressible nor motor cases that are relatively flexible because the iteration solution technique is poorly behaved.

The first difficulty lies in the equations of elasticity rather than the stiffness method. The equations of elasticity as usually stated are not valid for incompressible materials ($\nu = 0.5$). With displacements as unknowns, the equations contain coefficients that approach infinity as Poisson's ratio ν approaches 0.5. This behavior results in a deterioration of the numerical method as Poisson's ratio approaches 0.5.

A formulation suitable for incompressible isotropic nonhomogeneous materials is the following.

$$\text{Equilibrium:} \quad \sigma^{ij}{}_{,j} + f^i = 0 \quad , \tag{21}$$

$$\text{Strain-Displacement:} \quad \epsilon_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i}) \quad , \tag{22}$$

$$\text{Stress-Strain:} \quad \sigma_{ij} = \frac{P}{\left(1 + \frac{2}{3}\frac{\mu}{\lambda}\right)}\delta_{ij} + 2\mu(\epsilon_{ij} - aT\,\delta_{ij}) \quad , \tag{23}$$

$$\text{Mean Strain-Mean Stress:} \quad \epsilon^k_k - 3aT = \frac{P}{\lambda + \frac{2}{3}\mu} \quad , \tag{24}$$

where

σ^{ij} is the stress tensor,

ϵ_{ij} is the strain tensor,

u_i is the displacement vector,

λ, μ are the Lamé constants,

a   is the coefficient of thermal expansion

T   is the temperature change,

f^{α} is the body force vector,

p   is the mean stress or hydrostatic pressure.

The comma denotes covariant differentiation, and the repeated indices are summed. The indices range over the values 1, 2, and 3.

At each point of the boundary of the solid with unit normal n_i, either the displacement vector u_i or the surface force

$$F^i = \sigma^{ij} n_j \tag{25}$$

is specified. The temperature and body forces are given throughout the body.

This formulation is suitable for all values of Poisson's ratio in the interval $0 < \nu \leqslant 0.5$. It does not exhibit the infinite coefficients at $\nu = 0.5$ that the usual formulation does.

The stiffness method has been applied to this set of equations by Mathematical Sciences Corporation. The resulting program is documented in MSC Report No. 65-21-3.

The second difficulty occurring with the relatively flexible cases lies in the implementation of the iterative solution technique. In Figure 7 the solid and the shell are shown as two separate problems, much the way they are considered by the program. Also shown is an internal pressure load p .
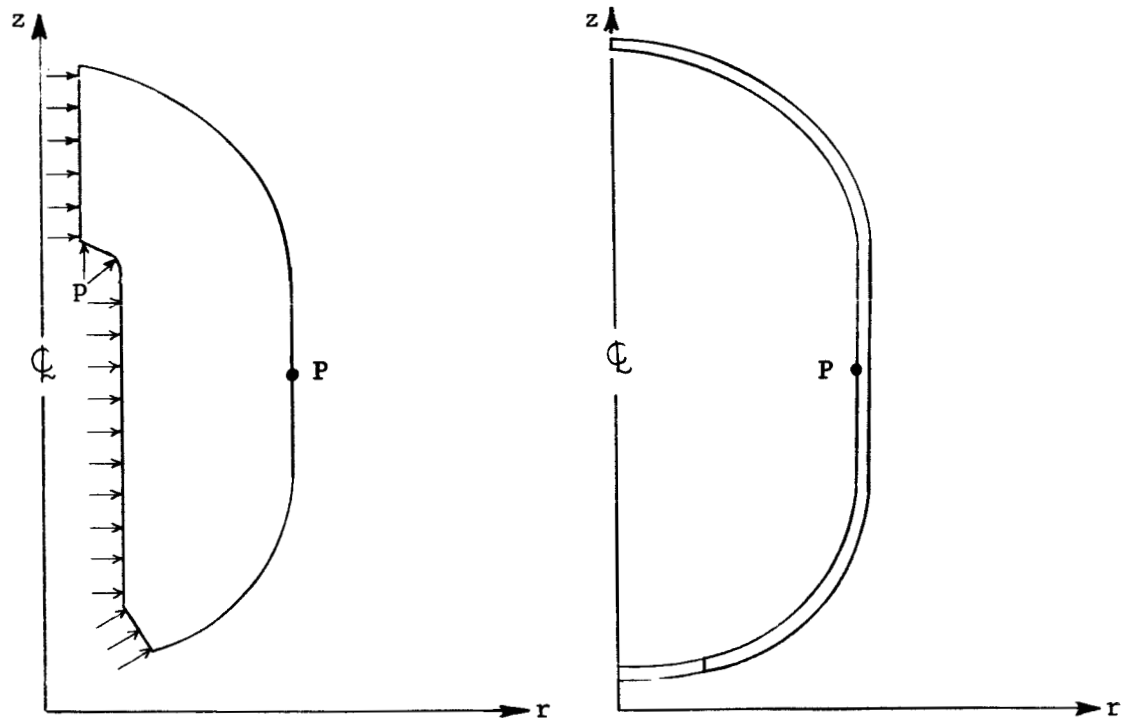

Figure 7

Consider the point P in Figure 7 and its radial displacement u . If the boundary of the solid is held rigidly, then there is no displacement but a reaction stress σ . If the boundary of the solid is free, then there is a displacement u but no stress σ . If the case experiences a stress σ , then there is a displacement u . If there is no stress on the case, then there is no displacement. These statements are repeated graphically in Figure 8.
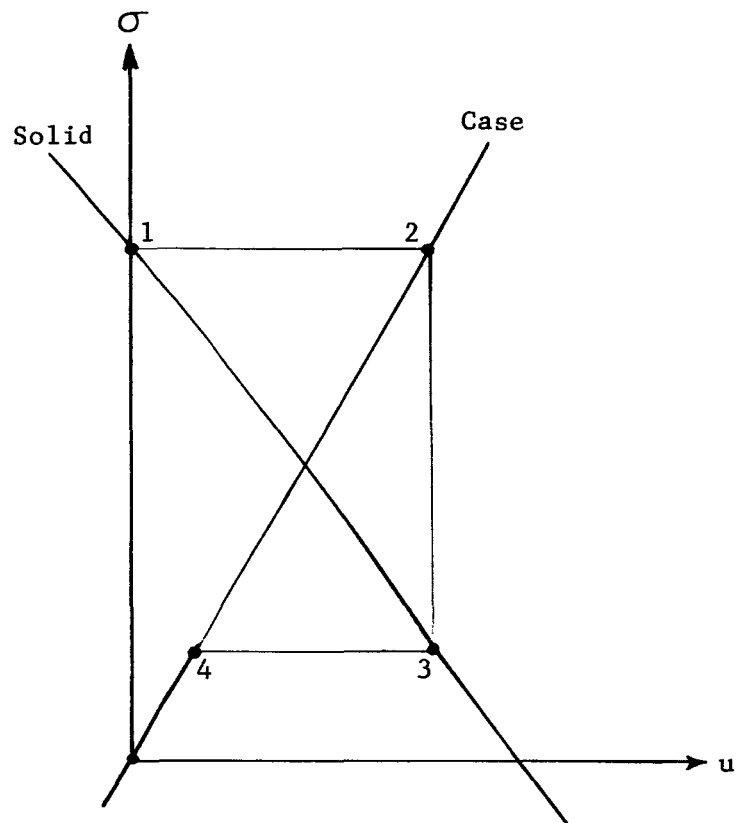
Figure 8

The solution to the combined problem is the intersection of case and solid curves. The iterative solution proceeds as follows. A solution for the solid is obtained using zero values for the shell displacements, point 1. This results in reaction loads σ_1 on the shell. The deflections of the shell are then obtained due to these reactions, point 2. The solid is held at the corrected displacements u_2 and a solution obtained, point 3. The procedure is then repeated until the error is as small as desired.

This iterative technique works very well for relatively stiff cases. A very high slope of the case curve in Figure 8 is a relatively stiff case; a flatter slope is a relatively flexible case. If this process is

repeated for flatter and flatter slopes of the case curve, a point will
be reached where the process no longer spirals inward but diverges.

By the time the solution at point 4 is established, there is suffi-
cient information available to estimate the intersection of the solid and
case curves. The program does this and continues. The estimation of the
intersection is repeated every time sufficient information is available.

For relatively stiff cases this process works very well. Below a
certain case flexibility the iterative process begins to deteriorate,
and the error cannot be reduced to a satisfactory level.

When the solid and the case stiffness matrices are combined and a
solution sought directly, just the opposite occurs. The relatively stiff
cases lead to numerical difficulties, while the relatively flexible cases
are the easiest to handle.

6. RECOMMENDATIONS FOR FUTURE WORK

The application of the stiffness method to the incompressible problem is well on its way to being an accomplished fact.

The iterative technique has proven, in many problems, to be a very effective means to incorporate the flexible case into the analysis. For the relatively flexible case, more work needs to be done with the iterative technique in order to expand the range of case flexibilities that may be handled. It is recommended that two avenues be followed. Only elementary numerical means have been used to find the intersection of the case and solid curves in Figure 8. Continued effort in this area should lead to useful improvements. Second, at some point it will become more profitable in terms of machine time and accuracy to combine the shell and case stiffness matrices and seek the solution directly. This should be included as a program option.

Although the application of the stiffness method to the incompressible equations and the considerations of a flexible case are separate ideas, their incorporation into a computer program cannot proceed independently. As a result, further study is required in order to incorporate the flexible cases into the incompressible programs.